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AN ALGORITHM OF THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS

PMM Vol. 32, №6, 1968, pp. 1089-1092

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(Received June 10, 1968)

Numerous problems of the theory of shells involve the solution of nonlinear boundary value problems [1 and 2] and this is often a fairly difficult task. Below we show, that in the number of cases numerical solutions of such problems are feasible.

Let us have the following system of differential equations with given boundary conditions:

$$dY_n(x)\,/\,dx = f_n(Y_n(x), q) \qquad (1)$$

$$\varphi_p(Y_n(0)) = 0 \quad \text{for} \quad x = 0, \qquad \psi_s(Y_n(1)) = 0 \quad \text{for} \quad x = 1 \qquad (2)$$

where

$$Y_n(x) = (y_1(x), \ldots, y_n(x)), \qquad f_n = (f_1, \ldots, f_n)$$

$$\varphi_p(Y_n(0)) = (\varphi_1(Y_n(0)), \ldots, \varphi_p(Y_n(0)))$$

$$\psi_s(Y_n(1)) = (\psi_1(Y_n(1)), \ldots, \psi_s(Y_n(1))), \qquad p + s = n \qquad (3)$$

Here q is a parameter, and the type of solution depends on the numerical value of this parameter.

Let us replace some of the conditions given in (3) by conditions formulated in an integral form, e. g.

$$\int_0^1 F(Y_n(x))\,dx + A = 0 \qquad (4)$$

and let us introduce the following auxilliary function:

$$y_{n+1}(x) = \int_0^x F(Y_n(x))\,dx + A \qquad (5)$$

If the integrand function is continuous on $x \in [0,1]$, we can write

$$dy_{n+1}/dx = F(Y_n(x)), \; y_{n+1}(0) = A, \qquad y_{n+1}(1) = 0 \qquad (6)$$

Considering now the problem in an $(n+1)$-dimensional space, we arrive at the problem which was formulated above.

Solution of the problem (1)–(3) is obtained as follows. Keeping $q = q_0$

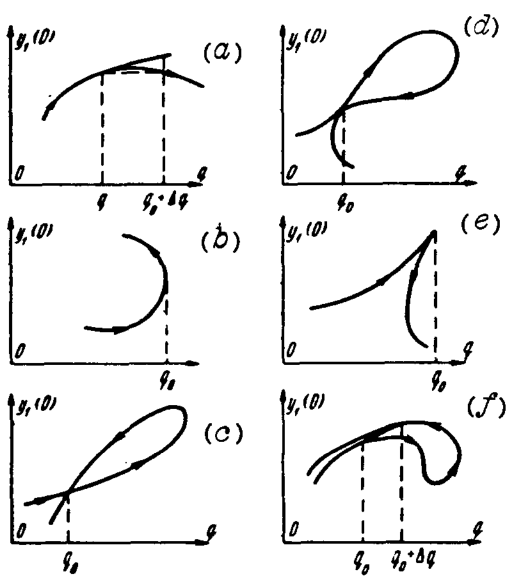

Fig. 1

fixed, we assign some values to the following unknown functions at $x = 0$:

$$y_{p+1}(0), \quad y_{p+2}(0), \ldots, y_n(0) \tag{7}$$

Inserting (7) into the first condition of (2), we obtain a system of equations in

$$y_1(0), \quad y_2(0), \ldots, y_p(0) \tag{8}$$

which, provided that definite conditions can be imposed on the function φ_p, can be solved using one of the approximate methods of solution of the systems of nonlinear equations [3].

Having obtained the values of the functions at $x = 0$, we can now integrate the system (1) using the Runge-Kutta or some other method, to obtain the vector $\psi_s(Y_n(1))$ which depends on the initial values assigned to the s unknowns in (7). Choosing these initial values so that

$$\psi_s(y_n(1)) = 0 \tag{9}$$

we obtain the required solution.

In this manner we have obtained s equations in s unknowns, and their functional dependence can be established with the help of the system (1). Although the latter cannot be written in an explicit form, we can solve it, using one of the approximate methods. Putting

$$q = q_0 + \Delta q \tag{10}$$

we seek its solution, using

$$y_n^*(0) = y_n^{**}(0) + \frac{dy_n(0)}{dq}\Delta q \quad \text{when} \quad x = 0 \tag{11}$$

as its first approximation. Here $y_n^*(0)$ denotes the approximate value of the vector for $q = q_0 + \Delta q$ when $0 = x$, and $y_n^{**}(0)$ is the value of the vector for $q = q_0$ when $x = 0$ (in the first step, Formula (11) is used without its second right-hand side term).

System (2) yields more accurate values of the unknowns (8) and the remaining steps are the same as those for $q = q_0$.

If the predetermined iteration number K is found to be insufficient to give the required accuracy, it means that the first approximation given by (11) was not good enough. This could, for example, happen in the case illustrated on the Fig. 1a. To ensure the convergence, the increment Δq should now be halved and the process repeated, and this should be done again, if required, until K iterations yield the required accuracy. When the "slow" region has been passed, the value of the increment Δq should be augmented.

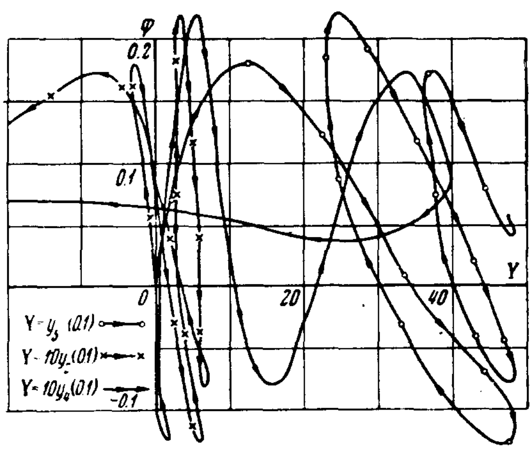

Fig. 2

During the process of computation, a check should be kept on the elements of the vector $dy_n(0)/dq$. Should any of these elements begin to increase excessively, then another fixed parameter should replace q. Thus, if the curve $(y_1(0), q)$ has a vertical tangent at the point q_0 (Fig. 1b), then $dy_1(0)/dq$ increases in the neighborhood of q_0. In this case the fixed q should be replaced with fixed $y_1(0)$ and the latter used as the parameter during the progression along the curve. The case when several components of the vector increase simultaneously during the computation, must be considered separately.

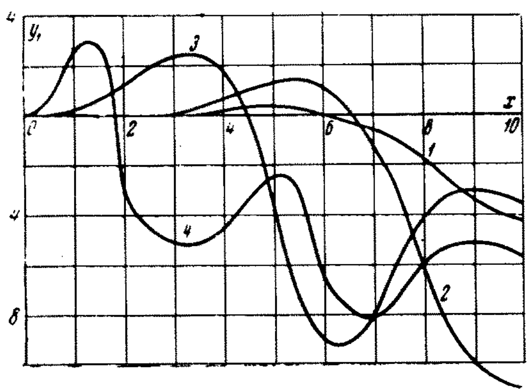

Fig. 3

Use of Formula (11) to obtain the initial approximation ensures that a loop of the type shown on the Fig. 1c is bypassed in such a manner, that the angle of inclination of the tangent to the curve varies continuously. More complex loops (Fig. 1d) require the incorporation of the higher order derivatives in (11), the order being related to the multiplicity of the point of contact of the branches. If singular points of the type shown on Fig. 1e present on the curve, then the problem requires special attention in the neighborhood of q_0.

When the curve passes near its other segment as shown on Fig. 1f, then the possibility arises of a jump from one part of the curve to the other.

The probability of such a transition diminishes, when (11) is supplemented with higher order terms and the increment in q is made smaller.

The solution becomes simpler, if the first set of conditions (2) is linear, or easily soluble in the unknowns (8). If this is true for the second set of conditions (2), then the integration should be performed from $x = 1$.

Brief analysis given above does not exhaust all the possibilities and examples can be found which might prove difficult to solve by the above methods, but the use of the algorithm enables us to obtain solutions of problems which are difficult to solve by other methods.

In particular, we have solved (the author together with V. N. Stegnii) the following nonlinear system which resulted from considering the equilibrium forms and deformation energy of hollow spherical shells

$$y_1' = y_2, \qquad y_2' = -\frac{y_2}{x} + \frac{y_1}{x^2} + y_3\left(1 + \frac{y_1}{x}\right) + 2qx$$

$$y_3' = y_4, \qquad y_4' = -\frac{y_4}{x} + \frac{y_3}{x^2} - y_1\left(1 + \frac{y_1}{2x}\right)$$

$$y_5' = y_1, \qquad y_6' = xy_2^2 + \frac{y_1^2}{x}, \qquad y_7' = xy_4^2 + \frac{y_3^2}{x}$$

$$y_2 - 10y_1 = 0, \qquad -y_6 + 0.01y_2^2 + 0.3y_1^2 = 0$$

$$y_4 - 10y_3 = 0, \ -y_7 + 0.01\, y_4^2 - 0.3y_3^2 = 0, \qquad \text{for } x = 0.1$$

$$y_2 + 0.03y_1 = 0, \qquad y_3 = y_5 = 0 \qquad \text{for } x = 10$$

Some of the results are shown on Fig. 2. Arrows on the curves indicate the direction of motion when the parameter q varies continuously from zero. We see that the problem has several solutions for each fixed q. Thus, for $q = 0.140$ the problem has four solutions. Fig. 3 shows the graphs of $y_1(x)$ corresponding to these solutions. Numbers accompanying the curves indicate the order in which they were obtained when q was varied continuously from zero.

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Translated by L. K.

ON THE LOSS OF STABILITY OF THE SHAPE OF AN IDEALLY FLEXIBLE STRING

PMM Vol. 32, №6, 1968, pp. 1092-1096

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(Received April 19, 1968)

The equations of the dynamics of an ideally flexible string were solved relative to the curvature and torsion of its shape in [1], and the characteristic wave propagation velocities of these parameters were found. A connection between the characteristic velocities and the loss of stability of the shape is established herein, which is identified with the loss in correctness of formulating problems with initial conditions.

We understand an ideally flexible string to be a material line which does not resist a change in shape, i. e. in curvature Ω_3 and torsion Ω_1.

Let the unperturbed motion of the string be characterized by the equations

$$\Omega_i^0 = \Omega_i^0(s, t) \qquad (i = 1,3)$$

Here s is the arc coordinate, and t the time.

Let us give some small deviations $\varepsilon_i^0(s)$ from the unperturbed values to the curvature and torsion by demanding that these deviations satisfy appropriate boundary conditions. The perturbed motion of the string then becomes

$$\Omega_i = \Omega_i^0(s, t) + \varepsilon_i(s, t) \qquad (i = 1,3)$$

In some domain $D\ (0 \leqslant s \leqslant s_1,\ 0 \leqslant t \leqslant t_1)$ let the following inequalities hold

$$\max|\Omega_i - \Omega_i^0| < \delta, \qquad \max|\varepsilon_i| < \nu \tag{1}$$

Let us consider the string shape unstable in the domain D if

$$\delta \to \delta_0 > 0 \text{ for } \nu \to 0. \tag{2}$$

Let us consider an arbitrary system of equations with constant coefficients

$$\sum_{j=1}^{n}\left(a_{lj}\frac{\partial q_j}{\partial t} + b_{lj}\frac{\partial q_j}{\partial s} + c_{lj}q_j\right) = 0 \tag{3}$$

Let the initial conditions be

$$q_m^\circ = \frac{1}{\lambda_1}e^{-\lambda_1 s i}, \quad \lambda_1 > 0; \quad q_j^\circ = 0, \quad (j = 1, 2, \ldots, n \neq m) \quad \text{for} \quad t = 0;\ 0 \leqslant s \leqslant s_1 \tag{4}$$

Let us assume that the relationships (4) satisfy the boundary conditions of Eq. (3).